Program
West Coast Algebraic Topology Summer School
September 6-8, 2013
The University of Oregon

FROM SPECTRAL ALGEBRA TO HIGHER ALGEBRA

1. Day 1

1.1. **Introductions (Dev Sinha, University of Oregon).** Welcome!

1.2. **An introduction to highly structured ring spectra (Henry Yi-Wei Chan, The University of Chicago).** What is a highly structured ring spectrum? Geometric constructions tend to give you highly structured ring spectra (cobordism, K-theory, ordinary homology). Highly structured ring spectra have extra operations (power operations, Dyer-Lashof operations). Simplicial sets and symmetric spectra: Explain why symmetric spectra are symmetric monoidal when ordinary spectra are not. Discuss why the weak equivalences are not homotopy isomorphisms but cohomology isomorphisms [HSS00, Sch07].

1.3. **An overview of model categories (Nerses Aramyan, University of Illinois Urbana-Champaign).** Construction of the homotopy category (though not in detail), cofibrant generation, examples, closed symmetric monoidal structure, Quillen equivalence [DS95, Hov99].

1.4. **Algebraic motivation: The abelian case (Jonathan Huang, University of Maryland).** Characterization of categories of modules as abelian categories with a compact projective generator (and exact direct limits). The basic setup of Morita theory. Related topics include the Eilenberg-Watts theorem and the characterization of Grothendieck categories. Start with [SS03] and the references therein.

1.5. **What is a stable model category? What is a spectral category? The Eilenberg-Watts theorem (Paul Van Koughnett, Northwestern University).** Definitions, examples. Every "nice enough" stable model category is equivalent to a spectral model category: The idea here is basically simple: Just form spectra on your stable model category. This will be Quillen equivalent because you are already stable, and it will be spectral if your original model category started out simplicial [Dug06]. Other theorems if there is interest: Morita theory is the easiest, but there are versions of Eilenberg-Watts and the Gabriel-Popescu theorem (about embedding of Grothendieck categories) as well [Hov09, SS03].

2. Day 2

2.1. **First version of Schwede-Shipley (David White, Wesleyan University).** If C is a spectral stable model category whose homotopy category has a small generator, then it is Quillen equivalent to modules over the endomorphism ring spectrum. This is Theorem 3.9.3 of Schwede-Shipley, specialized to the case where there is only one generator. Examples: Schwede-Shipley have many of them. Morita theory for ring spectra [SS03].

2.2. **Transition to higher categories: What is an $(\infty, 1)$ category? (Elden Elmanto, Northwestern University).** The passage from a simplicial model category to its associated infinity-category. Explain the basic idea that the homotopy category of a model category has entire mapping spaces- that is really the key aspect. So to remember all the extra structure we might as well just remember those; this leads to the easiest definition of an ∞-category as a topological category with a discrete space of objects. But this is not the definition Lurie adopts, so ease into that one [Ber06, Cam10, Lur08, Lur09, 1.1,1.2].

2.3. **Quasicategories (Joe Hannon, Boston University).** Nerve and realization of simplicial sets via Kan extension [Gro10, 1.17; Rie13, 1.5.5], Composition and horn filling [Tan12, Lur09, 1.1.2.2 and 1.1.2.3], Kan complexes = ∞-groupoids = spaces [Tan12, Lur09, 1.2.5], Quasi-categories admit composition up to a contractible space of choices [Joy08, Proposition 2.24], Morphism spaces [Tan12; Lur09, 1.2.2], Homotopy category of a quasi-category [Lur09, 1.2.3].

2.4. **Basics of ∞-category theory (Vitaly Lorman, The Johns Hopkins University).** Functors, Joins, Cones, Slices, Initial and Final Objects, (co)-Limits, Relation to homotopy (co)-limits in simplicial categories [Gro10, 2.1.6; Lur09, 4.2.4], Correspondences [Lur09, 5.2.1] , Adjunctions [Lur09, 5.2.2; Tan12].

2.5. **Stable ∞-categories (Robert Hank, The University of Minnesota).** Cover [Lur12, 1.1] Definition [Lur12, 1.1.1], The homotopy category of a stable ∞-category is triangulated (obviously do not check all of the axioms here) [Lur12, 1.1.2], Closure properties of the category of stable ∞-categories [Lur12, 1.1.3], Exact functors [Lur12, 1.1.4]. Section 1.3.4, beginning at 1.3.4.15: discussion of how to associate an ∞-category to a non-simplicial model category. Proof that a combinatorial model category gives a presentable ∞-category.

3. Day 3

3.1. **Stabilization (Jonathan Beardsley, The Johns Hopkins University).** Section 1.4 of [Lur09]: Corollary 1.4.4.6 (spectra are the free stable ∞-category on one object, just as abelian groups are the free abelian category on one object). The general concept of stabilization of an ∞-category: This begins in Definition 1.4.2.8, or more accurately in Notation 1.4.2.5 (or before that with the definition of excisive functors- note the connection to Goodwillie calculus here). The crucial facts are that the stabilization is stable (Corollary 1.4.2.17) and Proposition 1.4.2.21 (stabilizing an already stable thing does not change it). This is the reason that stable ∞-categories can be treated as modules over spectra: Because a stable ∞ category is equivalent to its stabilization and the stabilization is a module over spectra.

3.2. **The Barr-Beck theorem (Chris Kapulkin, University of Pittsburgh).** The Barr-Beck theorem, Theorem 6.2.2.5, Lemma 6.2.2.11 of [Lur12]. A way to approach this theorem is to prove the ordinary category version of it first [EE, Tan12]. Focus on the classical case and then state the changes that happen in the infinity case (presentations become simplicial objects). Prove that a co-continuous monad on an abelian category is a ring and use it as your running example. In particular you should show that algebras over the monad are the same as modules over the ring. Monads and adjunctions, algebras over a monad, the comparison map, Barr-Beck [EE, Tan12].

3.3. **The monoidal category of presentable stable ∞-categories (Justin Hilburn, University of Oregon).** Section 6.3.2 of [Lur12], which discusses the smash product monoidal structure on spectra. Obviously a lot of stuff has to be taken on faith to do this, but one might be able to capture the flavor this way. In particular, Proposition 6.3.2.11 is very nice; it essentially says that pointed presentable infinity categories are the same thing as modules over pointed simplicial sets. Remark 6.3.2.14. These lead to the corresponding results for spectra, Proposition 6.3.2.18 and the remarks that follow. [Lur12, 8.1.2.4].

3.4. **The Barr-Beck-Lurie theorem (Schwede-Shipley in ∞-categories) (Deb Vicinsky, University of Oregon).** Prove the Morita theorem using Barr-Beck [Mat]. The canonical resolution of an algebra over a monad, prove Barr-Beck-Lurie [Tan12], the proof of the Morita theorem from [Mat] still works [Lur12, 8.1.2.1]. The only facts you need about presentable stable infinity categories are that the adjoint functor theorem holds and that $\hom(C, \_)$ preserves filtered colimits when C is compact. [Lur12, 8.1.2.4].

References

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